A SPECIAL FORM OF A QUARTIC SURFACE.

By Prof. J. I. Hutchinson, Ithaca, N. Y.

The quartic surface which is the locus of the vertex of a cone passing through six given points* a, b, c, d, e, f, has for its equation†

$$K = \begin{vmatrix} e_1 f_1 x_2 x_3 x_4 & x_1 & e_1 & f_1 \\ e_2 f_2 x_1 x_3 x_4 & x_2 & e_2 & f_2 \\ e_3 f_3 x_1 x_2 x_4 & x_3 & e_3 & f_3 \\ e_4 f_4 x_1 x_2 x_3 & x_4 & e_4 & f_4 \end{vmatrix} = 0 .$$

The nodal points of the surface are $a, b, \ldots, f$. Their coordinates are respectively, $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, (e_1, e_2, e_3, e_4), (f_1, f_2, f_3, f_4).

Among the curves lying on the surface, those of especial interest are

(1) the 15 join lines of the nodes;

(2) the 10 lines, intersections of the 10 pairs of planes determined by the 6 nodes;

(3) the twisted cubic passing through the nodes.

The coordinates of the surface can be variously expressed as hyperelliptic functions of two variables. For example we may take

$$\begin{aligned} x_1 : x_2 : x_3 : x_4 &= c_{01}\vartheta_{01}\vartheta_3\vartheta_{02}\vartheta_{04} : c_2\vartheta_2\vartheta_1\vartheta_3\vartheta_{04} : c_{03}\vartheta_{03}\vartheta_1\vartheta_{02}\vartheta_{04} : c_4\vartheta_4\vartheta_1\vartheta_3\vartheta_{03} , \\ e_1 : e_2 : e_3 : e_4 &= c_{01}c_0c_{23}c_{34} : c_2c_5c_{12}c_{23} : c_{03}c_0c_{12}c_{14} : c_4c_5c_{14}c_{34} , \\ f_1 : f_2 : f_3 : f_4 &= c_{01}c_5c_{12}c_{14} : c_2c_0c_{14}c_{34} : c_{03}c_5c_{23}c_{34} : c_4c_0c_{12}c_{23} , \end{aligned}$$

where $\vartheta_\lambda = \vartheta_\lambda(u, v)$, u, and v being two linearly independent hyperelliptic integrals of the first kind, of the form

$$\int \frac{(\alpha z + \beta)\, dz}{\sqrt{a_0 z^6 + a_1 z^5 + \ldots + a_6}}, \text{ and } c_\lambda = \vartheta_\lambda(0, 0) .$$

If now the roots of the sextic $a_0 z^6 + \ldots + a_6$ are in involution it is well known that the hyperelliptic functions of u, v can be expressed in terms of elliptic functions of u and elliptic functions of v by means of a transformation of the second degree.‡

* See Caspary. Bulletin des Sciences Mathématiques, 1891, pg. 308, where a complete list of references is given. Also, Humbert. Journal de Mathématiques, 1893, pg. 466.

† Caspary. Comptes Rendus, t. 112, pg. 1356 (1891).

‡ Cf. my dissertation, On the Reduction of Hyperelliptic Functions ($p = 2$) to Elliptic Functions by a Transformation of the Second Degree. Chicago, 1895.

This leads to the following relations among the theta constants†

$$c_{23}^{\;2} = -c_{14}^{\;2}\,,\quad c_{02}^{\;2} = c_4^{\;2}\,,\quad c_{01}^{\;2} = c_2^{\;2}\,.$$

Bearing these relations in mind we observe that the points e and f depend on one another in such a way that if e be any point (e_1, e_2, e_3, e_4), f must be one of the two points $f' = (e_2, e_1, e_4, e_3)$, $f'' = (e_2, e_1, -e_4, e_3)$.

A geometrical consequence is that the six nodal points lie in involution on the twisted cubic determined by them. To prove this it is only necessary to show that the six points can be joined in pairs by three lines which generate a quadric surface containing the cubic.‡

The equation of the cone $a - bcdef$ is

$$U = \begin{vmatrix} c_2 f_2 x_3 x_4 & e_2 & f_2 \\ c_3 f_3 x_2 x_4 & e_3 & f_3 \\ e_4 f_4 x_2 x_3 & e_4 & f_4 \end{vmatrix} = 0\,,$$

and of the cone $b - acdef$, is

$$-V = \begin{vmatrix} e_1 f_1 x_3 x_4 & e_1 & f_1 \\ e_3 f_3 x_1 x_4 & e_3 & f_3 \\ e_4 f_4 x_1 x_3 & e_4 & f_4 \end{vmatrix} = 0\,.$$

These two cones intersect in the line ab and the twisted cubic. The quadric surface

$$Q = e_1 f_1 U + e_2 f_2 V = 0$$

contains the line cd, besides the intersection of U and V.

If now we take for f either one of the points f', or f'', Q is found to contain also the line ef. Its equation reduces to

$$(e_1 e_3 - e_2 e_4)(x_1 x_4 - x_2 x_3) - (e_1 e_4 - e_2 e_3)(x_1 x_3 - x_2 x_4) = 0\,.$$

Every generator of Q belonging to the same set as ab, cd, ef, meets the cubic in two points which are paired with each other in the involution.

The remaining portion of the intersection of Q with the quartic surface K is a curve of the second order which, it is easy to see, consists of two non-intersecting straight lines. For, the equation of the quartic can be put in the form

$$XY + ZW = 0$$

* Cf. Weber. Crelle, Bd. 84, pg. 353.

† See Reye. Geometrie der Lage, (zweite Auflage), zweiter Abtheilung, pg. 98. Leipzig, 1882.

where

$$X = (e_1e_4 - e_2e_3)(e_2x_3 - e_3x_2)(e_1x_4 - e_4x_1)$$
$$Y = x_1x_4 - x_2x_3$$
$$Z = (e_1e_3 - e_2e_4)(e_2x_4 - e_4x_2)(e_3x_1 - e_1x_3)$$
$$W = x_1x_3 - x_2x_4 .$$

Both Q and K, then, contain the intersection of Y and W,* viz, the lines ab, cd, and

$$(1) \begin{cases} x_1 - x_2 = 0 \\ x_3 - x_4 = 0 , \end{cases} \qquad (2) \begin{cases} x_1 + x_2 = 0 \\ x_3 + x_4 = 0 . \end{cases}$$

The lines (1) and (2) are new lines on the quartic additional to the 25 lines on the unspecialized surface. As they intersect the lines ab, cd, ef, they belong to a secondary set of generators of Q, and, therefore, each intersects the twisted cubic in one point. The two points d_1 and d_2, so determined are the double-points of the involution. To prove this, it is necessary to show that the primary generators of Q passing through d_1, and d_2 are tangent to the cubic.

The coordinates of d_1 are $(\lambda, \lambda, 1, 1)$ where $\lambda = \dfrac{e_1e_2(e_3 + e_4)}{e_3e_4(e_1 + e_2)}$. The plane

$$x_1 + x_2 - \lambda(x_3 + x_4) = 0$$

meets Q in the line (2), and the primary generator through d_1, which I will call γ_1. Writing Q in the form

$$(e_1e_3 - e_2e_4)[x_3(x_1 + x_2) - x_1(x_3 + x_4)] \\ - (e_1e_4 - e_2e_3)[x_4(x_1 + x_2) - [x_1(x_3 + x_4)] = 0$$

and substituting λ for the ratio $\dfrac{x_1 + x_2}{x_3 + x_4}$ we obtain

$$(e_1 + e_2)(e_3 - e_4)x_1 - \lambda[(e_1e_3 - e_2e_4)x_3 - (e_1e_4 - e_2e_3)x_4] = 0 ,$$

the equation of a plane also containing the line γ_1. But this is the equation of the tangent plane to the cone V at the point d_1. The line γ_1 is, therefore, tangent to the cubic. Similarly for the primary generator of Q through the point d_2.

CORNELL UNIVERSITY, *Aug.*, 1896.

---

* It is interesting to note that both Y and W meet K in 8 lines each.